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# Torus Actions and the Halperin-Carlsson Conjecture

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We report some results concerning the Halperin-Carlsson conjecture. This is obtained as a joint work with Y. Kamishima.

## 1. INTRODUCTION

Recently the real Bott tower and its generalization have been studied by several people ([3], [10], [14], [15], [8]). A real Bott manifold is originally defined to be the set of real points in the Bott manifold [6]. Among several characterizations by group actions, the Halperin-Carlsson conjecture is true for real Bott manifolds. The Halperin-Carlsson torus conjecture says that if there is an almost free torus action  $T^k$  on a closed  $n$ -manifold  $M$ , the following inequality holds:

$$(1) \quad 2^k \leq \sum_{j=0}^n b_j.$$

Here  $b_j = \text{rank } H_j(M; \mathbb{Z})$  is the  $j$ -th Betti number of  $M$ . See [16] for details and the references therein, see also [7].

Another characterization is that a real Bott manifold  $M$  is a euclidean space form (Riemannian flat manifold). It is conceivable whether the *Halperin-Carlsson conjecture* holds for compact euclidean space forms more generally.

By this motivation we revisit the Conner-Raymond's injective torus actions [5]. In this direction, we shall introduce *injective-splitting action* of a torus  $T^k$  on closed aspherical manifolds more generally. Our purpose is to prove the Halperin-Carlsson conjecture for such torus actions affirmatively.

## 2. INJECTIVE-SPLITTING ACTION

Let  $T^k$  be a  $k$ -dimensional torus ( $k \geq 1$ ). Given an *effective*  $T^k$ -action on a closed manifold  $M$ , the orbit map at  $x \in M$  is defined to be  $\text{ev}(t) = tx$  ( $\forall t \in T^k$ ). If we denote  $\pi_1(T^k) = H_1(T^k; \mathbb{Z}) = \mathbb{Z}^k$  and  $\pi_1(M) = \pi$ , then the map  $\text{ev}$  induces a homomorphism  $\text{ev}_\# : \mathbb{Z}^k \rightarrow \pi$  and  $\text{ev}_* : \mathbb{Z}^k \rightarrow H_1(M; \mathbb{Z})$  respectively.

According to the definition of Conner-Raymond [5], if  $\text{ev}_\#$  is *injective*, the action  $(T^k, M)$  is said to be *injective*. (Note that the definition is independent of the choice of the base point  $x \in M$  [11, Theorem 2.4.2, also Subsection 11.1].) Classically it is known that  $\text{ev}_\#$  is injective for closed *aspherical* manifolds [4].

Let  $(T^k, M)$  be an injective  $T^k$ -action on a closed manifold  $M$ . We see that  $\text{Im}(\text{ev}_\#) \leq C(\pi)$  where  $C(\pi)$  is the center of  $\pi$  (cf. [9]). Put  $\text{Im}(\text{ev}_\#) = \mathbb{Z}^k$ . Letting  $Q = \pi / \mathbb{Z}^k$ , there is a central group extension:

$$(2) \quad 1 \rightarrow \mathbb{Z}^k \rightarrow \pi \longrightarrow Q \rightarrow 1.$$

**Definition 2.1.** A  $T^k$ -action is said to be *injective-splitting* if there exists a finite index normal subgroup  $Q'$  of  $Q$  such that the induced extension splits;

$$\pi' = \mathbb{Z}^k \times Q'.$$

### 3. STATEMENTS AND RESULTS

**Theorem A.** Suppose that a closed manifold  $M$  admits an injective-splitting  $T^k$ -action. Then the following holds.

$$(3) \quad {}_k C_j \leq b_j.$$

In particular the Halperin-Carlsson conjecture is true.

On the other hand, if  $\text{ev}_* : \mathbb{Z}^k \rightarrow H_1(M; \mathbb{Z})$  is injective, then the  $T^k$ -action is said to be *homologically injective* (cf. [5]). Any homologically injective action is obviously injective.

**Proposition 3.1.** Any homologically injective  $T^k$ -action on a closed manifold  $M$  is injective-splitting.

*Proof.* The proof is essentially the same as [5, 2.2. Lemma]. Let  $1 \rightarrow \mathbb{Z}^k \rightarrow \pi \rightarrow Q \rightarrow 1$  be the central group extension. As  $\text{ev}_* : H_1(T^k; \mathbb{Z}) = \mathbb{Z}^k \rightarrow H_1(M; \mathbb{Z}) = \mathbb{Z}^\ell \oplus F$  is injective,  $\text{ev}_*(\mathbb{Z}^k) \leq \mathbb{Z}^\ell$  such that  $\text{ev}_*(\mathbb{Z}^k) \oplus \mathbb{Z}^{\ell-k} \leq \mathbb{Z}^\ell$ . If  $q : \pi \rightarrow H_1(M; \mathbb{Z})$  is a canonical projection, then  $\pi' = q^{-1}(\text{ev}_*(\mathbb{Z}^k) \oplus \mathbb{Z}^{\ell-k} \oplus F)$  is a finite index normal splitting subgroup of  $\pi$ .  $\square$

**Theorem B.** If  $T^k$  is a homologically injective action on a closed  $n$ -manifold  $M$ , then

$$(4) \quad {}_k C_j \leq b_j \quad (j = 0, \dots, k).$$

In particular the Halperin-Carlsson conjecture is true.

**Corollary B.** Every effective  $T^k$ -action on a compact  $n$ -dimensional euclidean space form  $M$  is injective-splitting. Thus  ${}_k C_j \leq b_j$ , the Halperin-Carlsson conjecture (1) holds.

We obtain a characterization of holomorphic torus actions originally observed by Carrell [2].

**Corollary C.** Every holomorphic action of the complex torus  $T_{\mathbb{C}}^k$  on a compact Kähler manifold is homologically injective. In particular,  ${}_{2k} C_j \leq b_j$ , the Halperin-Carlsson conjecture holds.

### 4. PRELIMINARIES FOR A PROOF OF THEOREM A

Suppose  $(T^k, M)$  is an injective action on a closed manifold  $M$ . Let  $\tilde{M}$  be the universal covering space of  $M$ . Since  $\mathbb{Z}^k \leq C(\pi)$ , letting  $Q = \pi/\mathbb{Z}^k$ , there is a central group extension:

$$(5) \quad 1 \rightarrow \mathbb{Z}^k \rightarrow \pi \rightarrow Q \rightarrow 1.$$

Now the universal covering group  $\mathbb{R}^k$  of  $T^k$  acts properly and freely on  $\tilde{M}$  such that  $\tilde{M} = \mathbb{R}^k \times W$  where  $W = \tilde{M}/\mathbb{R}^k$  is a simply connected smooth manifold. The central group extension (5) represents a 2-cocycle  $f$  in  $H^2(Q; \mathbb{Z}^k)$  in which  $\pi$  is viewed as the product  $\mathbb{Z}^k \times Q$  with group law:

$$(n, \alpha)(m, \beta) = (n + m + f(\alpha, \beta), \alpha\beta).$$

Let  $Map(W, \mathbb{R}^k)$  (respectively  $Map(W, T^k)$ ) be the set of smooth maps of  $W$  into  $\mathbb{R}^k$  (respectively  $T^k$ ) endowed with a  $Q$ -module structure in which there is an exact sequence of  $Q$ -modules [4]:

$$1 \rightarrow \mathbb{Z}^k \rightarrow Map(W, \mathbb{R}^k) \xrightarrow{\exp} Map(W, T^k) \rightarrow 1.$$

When  $Q$  acts properly discontinuously on  $W$  with compact quotient, we have the vanishing theorem from [4, Lemma 8.5], [11]:

$$(6) \quad H^i(Q, Map(W, \mathbb{R}^k)) = 0 \quad (i \geq 1).$$

By (6), the connected homomorphism induces an isomorphism :

$$\delta : H^1(Q; Map(W, T^k)) \rightarrow H^2(Q; \mathbb{Z}^k).$$

From this, there exists a map  $\chi : Q \rightarrow Map(W, \mathbb{R}^k)$  such that  $\delta^1 \chi = f$ . Then the action of  $\pi$  on  $\tilde{M}$  can be described as

$$(7) \quad \begin{aligned} (n, \alpha)(x, w) &= (n + x + \chi(\alpha)(\alpha w), \alpha w) \\ (\forall (n, \alpha) \in \pi, \forall (x, w) \in \mathbb{R}^k \times W). \end{aligned}$$

The  $\pi$ -action may depend on the choice of  $\chi'$  such that  $\delta^1 \chi' = f$ . However, the vanishing cohomology group (6) shows that

**Proposition 4.1.** *Such  $\pi$ -actions are equivalent to each other.*

## 5. PROOF OF THEOREM A

*Proof. Algebraic part.* (5) induces a commutative diagram:

$$(8) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \pi & \longrightarrow & Q \longrightarrow 1 \\ & & \parallel & & \uparrow \iota & & \uparrow \iota' \\ 1 & \longrightarrow & \mathbb{Z}^k & \longrightarrow & \pi' & \longrightarrow & Q' \longrightarrow 1 \end{array}$$

Here  $Q/Q'$  is a finite group by Definition 2.1. For the cocycle  $f$  representing the upper group extension, it follows  $\iota'^*[f] = 0 \in H^2(Q'; \mathbb{Z}^k)$  by the hypothesis. We may assume

$$(9) \quad f|_{Q'} = 0.$$

On the other hand, if  $\tau : H^2(Q'; \mathbb{Z}^k) \rightarrow H^2(Q; \mathbb{Z}^k)$  is the transfer homomorphism, then  $\tau \circ \iota'^* = |Q : Q'| : H^2(Q; \mathbb{Z}^k) \rightarrow H^2(Q; \mathbb{Z}^k)$  so that  $[f]$  is a torsion in  $H^2(Q; \mathbb{Z}^k)$ . There exists an integer  $\ell$  such that  $\ell \cdot f = \delta^1 \tilde{\lambda}$  for some function  $\tilde{\lambda} : Q \rightarrow \mathbb{Z}^k$ . Put  $\lambda = \frac{\tilde{\lambda}}{\ell} : Q \rightarrow \mathbb{R}^k$ . Then

$$(10) \quad f = \delta^1 \lambda.$$

The equation (9) shows  $[\lambda|_{Q'}] \in H^1(Q; \mathbb{R}^k)$ . Viewed  $\mathbb{R}^k \leq Map(W, \mathbb{R}^k)$  as constant maps,  $[\lambda|_{Q'}] \in H^1(Q; Map(W, \mathbb{R}^k)) = 0$  by (6). So there is an element  $h \in Map(W, \mathbb{R}^k)$  such that  $\lambda|_{Q'} = \delta^0 h$ . The equality  $\lambda(\alpha') = \delta^0 h(\alpha')(w)$  ( $\forall \alpha' \in Q', \forall w \in W$ ) implies

$$(11) \quad h(w) = h(\alpha'w) + \lambda(\alpha').$$

**Geometric part.** Noting Proposition 4.1, the  $\pi$ -action (7) on  $\tilde{M}$  is equivalent with

$$(12) \quad (n, \alpha)(x, w) = (n + x + \lambda(\alpha), \alpha w) \quad (\forall (x, w) \in \mathbb{R}^k \times W).$$

Recall that  $\pi$  has the splitting subgroup  $\pi' = \mathbb{Z}^k \times Q'$ . Obviously we have the product action of  $\mathbb{Z}^k \times Q'$  on  $\mathbb{R}^k \times W$  such that  $\mathbb{R}^k \times W / \mathbb{Z}^k \times Q' = T^k \times W / Q'$ . Define a diffeomorphism  $\tilde{G} : \mathbb{R}^k \times W \rightarrow \mathbb{R}^k \times W$  to be  $\tilde{G}(x, w) = (x + h(w), w)$ . Using (11), it is easy to check that  $\tilde{G} : (\pi', \mathbb{R}^k \times W) \rightarrow (\mathbb{Z}^k \times Q', \mathbb{R}^k \times W)$  is an equivariant diffeomorphism with respect to the action (12) and the product action. Putting  $\mathbb{R}^k \times W / \pi' = T^k \times W / Q'$  as a quotient space,  $\tilde{G}$  induces a diffeomorphism  $G : T^k \times W / Q' \rightarrow T^k \times W / Q'$ . Let  $q : T^k \times W \rightarrow T^k \times W / Q'$  be the covering map ( $q(t, w) = [t, w]$ ). Then

$$(13) \quad G \circ q(t, w) = G([t, w]) = (t \exp 2\pi i h(w), [w]).$$

Noting (12),  $\pi$  induces an action of  $Q$  on  $\tilde{M} / \mathbb{Z}^k = T^k \times W$  such that

$$(14) \quad \alpha(t, w) = (t \exp 2\pi i \lambda(\alpha), \alpha w) \quad (\forall \alpha \in Q).$$

$F = Q / Q'$  has an induced action on  $T^k \times W / Q'$  by  $\hat{\alpha}[t, w] = [t \exp 2\pi i \lambda(\alpha), \alpha w]$  ( $\forall \hat{\alpha} \in F$ ) which gives rise to a covering map:

$$(15) \quad F \rightarrow T^k \times W / Q' \xrightarrow{\nu} T^k \times W / Q = M.$$

For any  $\alpha \in Q$ , consider the commutative diagram:

$$(16) \quad \begin{array}{ccc} H_j(T^k \times W) & \xrightarrow{\alpha_*} & H_j(T^k \times W) \\ \downarrow q_* & & \downarrow q_* \\ H_j(T^k \times_{Q'} W) & \xrightarrow{\hat{\alpha}_*} & H_j(T^k \times_{Q'} W) \end{array}$$

in which  $H_j(T^k) \otimes H_0(W) \leq H_j(T^k \times W)$ . By the formula (14), the  $Q$ -action on the  $T^k$ -summand is a translation by  $\exp 2\pi i \lambda(\alpha) \in T^k$  so the homology action  $\alpha_*$  on  $H_j(T^k) \otimes H_0(W)$  is trivial. If  $H_j(T^k \times_{Q'} W)^F$  denotes the subgroup left fixed under the homology action for every element  $\hat{\alpha} \in F$ , it follows

$$(17) \quad q_*(H_j(T^k) \otimes H_0(W)) \leq H_j(T^k \times_{Q'} W)^F.$$

Using the transfer homomorphism,  $\nu$  of (15) induces an isomorphism:

$$\nu_* : H_j(T^k \times_{Q'} W; \mathbb{Q})^F \longrightarrow H_j(M; \mathbb{Q}).$$

In particular,  $\nu_* : q_*(H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q})) \rightarrow H_j(M; \mathbb{Q})$  is injective.

On the other hand, let  $q' : W \rightarrow W / Q'$  be the projection  $q'(w) = [w]$ . Define a homotopy  $\Psi_\theta : T^k \times W \rightarrow T^k \times W / Q'$  ( $\theta \in [0, 1]$ ) to be

$$\Psi_\theta(t, w) = (t \exp 2\pi i (\theta \cdot h(w)), [w]).$$

Then  $\Psi_0 = \text{id} \times q' \simeq G \circ q$  from (13). As  $G_* \circ q_* = \text{id} \times q'_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \rightarrow H_j(T^k; \mathbb{Q}) \otimes H_0(W / Q'; \mathbb{Q})$  is obviously isomorphic, it implies that  $q_* : H_j(T^k; \mathbb{Q}) \otimes$

$H_0(W; \mathbb{Q}) \longrightarrow H_j(T^k \times_{Q'} W; \mathbb{Q})$  is injective. If  $p = \nu \circ q : T^k \times W \rightarrow M$  is the projection, then  $p_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \longrightarrow H_j(M; \mathbb{Q})$  becomes injective. This shows Theorem A.  $\square$

## 6. APPLICATION TO EUCLIDEAN SPACE FORMS

Let  $M$  be a compact euclidean space form  $\mathbb{R}^n/\pi$  with  $\text{rank } H_1(M) = k$ , and set  $s = \text{rank } C(\pi)$ . In [5, § 7], Conner and Raymond stated (without proof) that Calabi's theorem [1] shows the existence of a  $T^k$ -action. From this, we see that  $k \leq s$  because  $\mathbb{Z}^k \leq C(\pi)$ . On the other hand, using the algebraic hull argument, it is easy to see that  $M$  admits an effective  $T^s$ -action, so by Corollary B,  $s \leq k$ . Therefore, we obtain:

**Theorem E.** *A compact  $n$ -dimensional euclidean space form  $M$  admits an action of  $T^k$ , where  $k = \text{rank } H_1(M)$ , in which  $\text{rank } C(\pi) = \text{rank } H_1(M)$ .*

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